ISOMETRIES BETWEEN THE SPACES OF L^1 HOLOMORPHIC QUADRATIC DIFFERENTIALS ON RIEMANN SURFACES OF FINITE TYPE

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Abstract

By applying the methods of V. Markovic [7] to the special case of Riemann surfaces of finite type, we obtain a transparent new proof of a classical result about isometries between the spaces of L^1 holomorphic quadratic differentials on such surfaces.

1. Introduction

It was already known to S. Banach that if $1 \leq p < \infty$ and $p \neq 2$, then every isometry of $L^p([0,1],\mathbb{R})$ into itself has the form $f \mapsto (f \circ \psi) \times h$, where $h \in L^p([0,1],\mathbb{R})$ and ψ is a Borel measurable map of $[0,1]$ into itself. (See [1, Chapter 11, §5] or, for more details, [10, Chapter 15, §7]; for an analogous result in more general L^p-spaces, see [6].)

More recently, there has been considerable interest in studying the isometries of various subspaces of L^p. The following very general result about $\mathbb{C}$-linear isometries between finite-dimensional subspaces of complex L^p-spaces was proved by W. Rudin in [11], where it is used to determine the isometries of certain H^p-spaces.

PROPOSITION 1 (Rudin [11, Theorem 1])
Assume that $0 < p < \infty$ and that p is not an even integer. Let μ and ν be finite positive measures on the sets X and Y. Let k be a positive integer, and let $f_1, \ldots, f_k$ in $L^p(\mu,\mathbb{C})$ and $g_1, \ldots, g_k$ in $L^p(\nu,\mathbb{C})$ satisfy

$$\int_X \Big|1 + \sum_{j=1}^{k} \lambda_j f_j\Big|^p \, d\mu = \int_Y \Big|1 + \sum_{j=1}^{k} \lambda_j g_j\Big|^p \, d\nu \quad \text{for all } (\lambda_1, \ldots, \lambda_k) \text{ in } \mathbb{C}^k. \tag{1}$$

If $F = (f_1, \ldots, f_k)$ and $G = (g_1, \ldots, g_k)$, then the maps $F\colon X \to \mathbb{C}^k$ and $G\colon Y \to \mathbb{C}^k$ satisfy

$$\mu\big(F^{-1}(E)\big) = \nu\big(G^{-1}(E)\big) \quad \text{for every Borel set } E \text{ in } \mathbb{C}^k. \tag{2}$$

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In [7] V. Markovic used the $p = 1$ case of Proposition 1 to show that in general every surjective $\mathbb{C}$-linear isometry between the spaces of L^1 holomorphic quadratic differentials on Riemann surfaces X and Y is related to a bijective holomorphic map between X and Y. A precise statement of this result, together with some background information and definitions, is given in §2.

In the general case considered in [7], applying Proposition 1 is not a straightforward task. Equation (2) is needed for every positive value of k, and it leads only to a biholomorphic map between open sets of full measure in X and Y. Extending this map to a biholomorphic map between X and Y requires considerable effort (see [7]).

In this paper we apply the methods of [7] to Riemann surfaces of finite type. We do this for two reasons. First, the resulting proof is shorter than the previously known proofs of this classical special case. Second, the methods of [7] become transparent in this setting. As we shall see, a single application of (2) leads to the required map between X and Y, with no need for analytic continuation.

2. Definitions and statement of Theorem 1

Let X be a Riemann surface. Recall that a quadratic differential on X is a tensor φ whose restriction to the domain U of any coordinate chart z has the form $f\,dz^2$, where f is a measurable function on U. If all these functions f are meromorphic (or holomorphic), φ is said to be meromorphic (or holomorphic).

For any quadratic differential φ on X, we denote by $|\varphi|$ the differential 2-form on X that equals $|f|\,dx\,dy$ in U if φ equals $f\,dz^2$ there. We say φ is integrable if its L^1-norm

$$\|\varphi\| = \iint_X |\varphi|$$

is finite. With the L^1-norm, the space of integrable quadratic differentials on X becomes a complex Banach space, and the integrable holomorphic quadratic differentials form a closed subspace that we denote by $Q(X)$.

Let φ be a quadratic differential on X, and let $h\colon Y \to X$ be a holomorphic map of a Riemann surface Y into X. By definition, the pullback of φ by h is the quadratic differential $h^*(\varphi)$ on Y that equals $(f \circ h)(d(z \circ h))^2$ in $h^{-1}(U)$ whenever φ equals $f\,dz^2$ in $U(\subset X)$.

If the holomorphic map $h\colon Y \to X$ is bijective, then $h^*(\varphi) \in Q(Y)$ for all φ in $Q(X)$, and the map

$$\varphi \mapsto h^*(\varphi), \quad \varphi \text{ in } Q(X),$$

is a $\mathbb{C}$-linear isometry of $Q(X)$ onto $Q(Y)$. It is shown in Markovic [7] that almost all surjective $\mathbb{C}$-linear isometries between spaces $Q(X)$ and $Q(Y)$ are essentially of that form. A precise statement requires two additional definitions.

We say that the Riemann surface X has finite type if there are a compact Riemann surface $\widehat{X}$ and a finite subset E of $\widehat{X}$ such that X is conformally equivalent to $\widehat{X} \setminus E$. If X is such a surface, the genus g of $\widehat{X}$ and the number $n\,(\geq 0)$ of points in E are determined by X. We say that the type of X is exceptional if $2g + n \leq 4$.

Now we can give a precise statement of the main theorem of [7].

PROPOSITION 2 (see Markovic [7, Theorem 1.1])
Let X and Y be Riemann surfaces that are not of exceptional finite type, and let $T\colon Q(X) \to Q(Y)$ be a surjective $\mathbb{C}$-linear isometry. There are a bijective holomorphic map $h\colon Y \to X$ and a complex number c such that $|c| = 1$ and $T(\varphi) = c\,h^*(\varphi)$ for all φ in $Q(X)$.

Special cases of Proposition 2 were proved some time ago (see, e.g., H. Royden [9] and N. Lakic [5]), but the method of proof in [7] is quite different. We use it here to give a simple new proof of the following special case of Proposition 2, first proved in C. Earle and I. Kra [3].

THEOREM 1
Suppose that X and Y are Riemann surfaces of nonexceptional finite type and that $T\colon Q(X) \to Q(Y)$ is a surjective $\mathbb{C}$-linear isometry. There are a bijective holomorphic map $h\colon Y \to X$ and a complex number c such that $|c| = 1$ and

$$T(\varphi) = c\,h^*(\varphi) \quad \textit{for all } \varphi \textit{ in } Q(X). \tag{3}$$

Our proof of Theorem 1 is given in §4, where we also compare it with the proof in [3] (see the closing remark).

3. Some projective embeddings of compact Riemann surfaces

Our proof requires some classical results from the theory of compact Riemann surfaces, which we review in this section for the reader’s convenience.

Let $\widehat{X}$ be a compact Riemann surface of genus $g \geq 0$, and let E be a finite, possibly empty, subset of $\widehat{X}$ which contains exactly $n \geq 0$ points. We assume that $2g + n \geq 5$, so that the Riemann surface $X = \widehat{X} \setminus E$ has nonexceptional finite type.

Each φ in the vector space $Q(X)$ can be regarded as a quadratic differential on $\widehat{X}$ which is holomorphic except for isolated singularities at the points of E. The integrability of $|\varphi|$ implies that these singularities are either removable or simple poles (see S. Nag [8, §14.13]), so $Q(X)$ is the vector space of meromorphic quadratic differentials on $\widehat{X}$ whose poles (if any) are simple and belong to E.

PROPOSITION 3

If $X = \widehat{X} \setminus E$ *as above, then* $\dim(Q(X)) = 3g - 3 + n \geq 2$. *If* $x \in E$, *then some* φ *in* $Q(X)$ *has a pole at* x.

Proof
This well-known result follows readily from O. Forster [4, Theorems 17.16, 17.19]. To see this, we use some classical machinery that can be found conveniently in [4, §§16, 17].

By definition, a divisor on $\widehat{X}$ is a function $D\colon \widehat{X} \to \mathbb{Z}$ such that $D(x) \neq 0$ for only finitely many points x in $\widehat{X}$. If D is a divisor on $\widehat{X}$, its degree is the number

$$\deg(D) = \sum_{x \in \widehat{X}} D(x).$$

For example, the characteristic function χ_E of E is a divisor on $\widehat{X}$ of degree n.

If α is a meromorphic function, one-form, or quadratic differential on $\widehat{X}$ and x is a point of $\widehat{X}$, we define the order of α at x to be

$$\operatorname{ord}_x(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is holomorphic and nonzero at } x, \\ k & \text{if } \alpha \text{ has a zero of order } k \text{ at } x, \\ -k & \text{if } \alpha \text{ has a pole of order } k \text{ at } x, \\ +\infty & \text{if } \alpha \text{ is identically zero on } \widehat{X}. \end{cases}$$

We say that α is trivial if it is identically zero on $\widehat{X}$. If α is nontrivial, then the map $x \mapsto \operatorname{ord}_x(\alpha)$ is a divisor on $\widehat{X}$, called the divisor of α and denoted by (α).

Let $\mathscr{M}(\widehat{X})$ be the field of meromorphic functions on $\widehat{X}$. For any divisor D on $\widehat{X}$, we define $\mathscr{O}_D(\widehat{X})$ to be the complex vector space of all the functions in $\mathscr{M}(\widehat{X})$ (including the zero function) that are multiples of the divisor $-D$:

$$\mathscr{O}_D(\widehat{X}) = \bigl\{ f \in \mathscr{M}(\widehat{X}) : \operatorname{ord}_x(f) \geq -D(x) \ \text{ for all } x \text{ in } \widehat{X} \bigr\}.$$

Now we can proceed. Let ω be a nontrivial meromorphic one-form on $\widehat{X}$. Consider the divisor $D = 2(\omega) + \chi_E$ on $\widehat{X}$. The space $Q(X)$ obviously consists precisely of the meromorphic quadratic differentials $\varphi = f\omega^2$ such that $f \in \mathscr{O}_D(\widehat{X})$.

The divisor of ω has degree $2g - 2$, so $D = 2(\omega) + \chi_E$ has degree $4g - 4 + n$. Since $2g + n \geq 5$ by assumption, we have $\deg(D) \geq 2g + 1$. Consequently, by the Riemann-Roch theorem and [4, Theorem 17.16],

$$\dim\bigl(Q(X)\bigr) = \dim\bigl(\mathscr{O}_D(\widehat{X})\bigr) = \deg(D) + 1 - g = 3g - 3 + n \geq g + 2 \geq 2.$$

That proves the first assertion of the proposition.

To complete the proof, choose any x in E. Since $\deg(D) \geq 2g + 1$, we can use [4, Theorem 17.19] to obtain f in $\mathscr{O}_D(\widehat{X})$ with $\operatorname{ord}_x(f) = -D(x)$. By the definition of D, the quadratic differential $f\omega^2$ in $Q(X)$ has a simple pole at x. □

Next we discuss projective embeddings of $\widehat{X}$ associated with $Q(X)$. Let k be a positive integer, and let $\mathbb{P}^k$ be the k-dimensional complex projective space. Each point $(z_0, \dots, z_k)$ in $\mathbb{C}^{k+1} \setminus \{0\}$ determines a point $[(z_0, \dots, z_k)]$ in $\mathbb{P}^k$. The formula

$$\pi_0(z_1, \dots, z_k) = \big[(1, z_1, \dots, z_k)\big], \quad (z_1, \dots, z_k) \in \mathbb{C}^k,$$

defines a biholomorphic map of $\mathbb{C}^k$ onto a dense open subset of $\mathbb{P}^k$.

PROPOSITION 4
Let $X = \widehat{X} \setminus E$ as above, and let $\varphi_0, \dots, \varphi_k$ be a basis for $Q(X)$. Set $f_j = \varphi_j/\varphi_0$, $j = 1, \dots, k$, and set

$$X_0 = \widehat{X} \setminus \big\{x \in \widehat{X} : \textit{some } f_j \textit{ has a pole at } x\big\}.$$

Let $F\colon X_0 \to \mathbb{C}^k$ be the holomorphic map $F = (f_1, \dots, f_k)$. There is a (unique) holomorphic embedding $\Phi : \widehat{X} \to \mathbb{P}^k$ such that

$$\Phi(x) = \pi_0\big(F(x)\big) \quad \textit{for all } x \textit{ in } X_0. \tag{4}$$

Proof
This result is a special case of [4, Theorem 17.22]. To see this, we consider the divisor $D = (\varphi_0) + \chi_E$ on $\widehat{X}$. Obviously, $Q(X)$ is the set of meromorphic quadratic differentials $\varphi = f\varphi_0$ such that $f \in \mathscr{O}_D(\widehat{X})$.

Again we have $\deg(D) \geq 2g + 1$. Since the functions $1, f_1, \dots, f_k$ are a basis for $\mathscr{O}_D(\widehat{X})$, we can use [4, Theorem 17.22] to conclude that the map

$$x \mapsto \big[(1, f_1(x), \dots, f_k(x))\big], \quad x \in \widehat{X},$$

when interpreted appropriately at the poles of the f_j, defines a holomorphic embedding of $\widehat{X}$ in $\mathbb{P}^k$. □

COROLLARY 1
The map F defined in Proposition 4 is a homeomorphism of X_0 onto a closed subset of $\mathbb{C}^k$.

Proof
Since F is holomorphic on X_0, it is continuous. Since $\Phi^{-1} \circ \pi_0 \circ F$ is the identity map of X_0 to itself, F is a homeomorphism.

To see that $F(X_0)$ is a closed set, consider a sequence (x_n) in X_0 such that $F(x_n)$ converges to a point $z = (z_1, \dots, z_k)$ in $\mathbb{C}^k$. We may assume that x_n converges to some point $\widehat{x}$ in $\widehat{X}$. Then $f_j(\widehat{x}) = z_j \neq \infty$, $j = 1, \dots, k$, so $\widehat{x}$ belongs to X_0 and $z = F(\widehat{x})$. □

4. Proof of Theorem 1

We may assume that the given Riemann surfaces X and Y are the complements of finite sets in compact Riemann surfaces $\widehat{X}$ and $\widehat{Y}$, respectively. We do not assume that $\widehat{X}$ and $\widehat{Y}$ have the same genus.

Let $\varphi_0, \dots, \varphi_k$ be a basis for $Q(X)$, and define X_0 and the map $F = (f_1, \dots, f_k)$ as in Proposition 4.

Set $\psi_j = T(\varphi_j)$, $j = 0, \dots, k$. Since $T\colon Q(X) \to Q(Y)$ is a surjective $\mathbb{C}$-linear isometry, $\psi_0, \dots, \psi_k$ is a basis for $Q(Y)$. Set $g_j = \psi_j/\psi_0$, $j = 1, \dots, k$, and set

$$Y_0 = \widehat{Y} \setminus \big\{y \in \widehat{Y} : \text{some } g_j \text{ has a pole at } y\big\}.$$

Let $G\colon Y_0 \to \mathbb{C}^k$ be the holomorphic map $G = (g_1, \dots, g_k)$. By Proposition 4, there is a holomorphic embedding $\Psi\colon \widehat{Y} \to \mathbb{P}^k$ such that $\Psi = \pi_0 \circ G$ on Y_0.

Let μ and ν be the finite positive Borel measures on X_0 and Y_0 defined by

$$\mu(A) = \iint_A |\varphi_0| \qquad \text{and} \qquad \nu(B) = \iint_B |\psi_0|$$

for all Borel sets A in X_0 and B in Y_0. Since T is a $\mathbb{C}$-linear isometry, we have

$$\begin{aligned}\iint_{X_0} \Big|1 + \sum_{j=1}^{k} \lambda_j f_j\Big|\, d\mu &= \iint_X \Big|\varphi_0 + \sum_{j=1}^{k} \lambda_j \varphi_j\Big| \\ &= \iint_Y \Big|\psi_0 + \sum_{j=1}^{k} \lambda_j \psi_j\Big| = \iint_{Y_0} \Big|1 + \sum_{j=1}^{k} \lambda_j g_j\Big|\, d\nu\end{aligned}$$

for all $(\lambda_1, \dots, \lambda_k)$ in $\mathbb{C}^k$. Therefore, by Proposition 1, maps F and G and measures μ and ν satisfy Rudin's equimeasurabilty condition (2). Applying (2) to the closed subset $F(X_0)$ of $\mathbb{C}^k$, we obtain

$$\begin{aligned}\|\varphi_0\| &= \iint_{X_0} |\varphi_0| = \mu(X_0) = \nu\big(G^{-1}(F(X_0))\big) \\ &= \iint_{G^{-1}(F(X_0))} |\psi_0| \le \iint_{Y_0} |\psi_0| = \|\psi_0\|. \end{aligned} \tag{5}$$

Since $\|\psi_0\| = \|\varphi_0\|$, the weak inequality in (5) is an equality, and $G^{-1}(F(X_0))$ has full measure in Y_0. Since it is a closed subset of Y_0, $G^{-1}(F(X_0))$ equals Y_0, and $G(Y_0)$ is contained in $F(X_0)$.

Similarly, by applying (2) to the set $G(Y_0)$, we find that $F(X_0)$ is a subset of $G(Y_0)$. Therefore the sets $F(X_0)$ and $G(Y_0)$ are equal, and so are their images under the map π_0 from $\mathbb{C}^k$ to $\mathbb{P}^k$. Now $\pi_0(F(X_0)) = \Phi(X_0)$ is dense in the compact set $\Phi(\widehat{X})$, and $\pi_0(G(Y_0))$ is dense in the compact set $\Psi(\widehat{Y})$, so the sets $\Phi(\widehat{X})$ and $\Psi(\widehat{Y})$ are equal.

Let $h\colon \widehat{Y} \to \widehat{X}$ be the bijective holomorphic map $\Phi^{-1} \circ \Psi$. The restriction of h to Y_0 satisfies $F \circ h = G$ and $h(Y_0) = X_0$. From the definitions of F and G, we obtain

$$\frac{T(\varphi_j)}{T(\varphi_0)} = \frac{\psi_j}{\psi_0} = g_j = f_j \circ h = \frac{\varphi_j}{\varphi_0} \circ h = \frac{h^*(\varphi_j)}{h^*(\varphi_0)}, \quad j = 1, \dots, k,$$

so

$$\frac{T(\varphi)}{T(\varphi_0)} = \frac{h^*(\varphi)}{h^*(\varphi_0)} \quad \text{for all } \varphi \text{ in } Q(X). \tag{6}$$

Let K be any compact set in Y_0. Applying equation (2) to the compact set $G(K)$ in $\mathbb{C}^k$, we find that

$$\iint_K |T(\varphi_0)| = \iint_K |\psi_0| = \nu(K) = \mu\big(h(K)\big) = \iint_{h(K)} |\varphi_0| = \iint_K |h^*(\varphi_0)|.$$

Since K is arbitrary, we must have $|T(\varphi_0)| = |h^*(\varphi_0)|$ in Y_0 and hence in all of $\widehat{Y}$. Therefore $T(\varphi_0) = e^{i\theta} h^*(\varphi_0)$ for some real number θ, and (6) becomes

$$T(\varphi) = e^{i\theta} h^*(\varphi) \quad \text{for all } \varphi \text{ in } Q(X). \tag{7}$$

To complete the proof, we must show that $h(Y) = X$. By Proposition 3, Y is the set of points in $\widehat{Y}$ where every $T(\varphi)$ is finite, and $h^{-1}(X)$ is the set of points in $\widehat{Y}$ where every $h^*(\varphi)$ is finite. Equation (7) says that these sets coincide. □

Remark. It is shown in [3] that if X and Y have nonexceptional finite type, then any surjective $\mathbb{R}$-linear isometry from $Q(X)$ to $Q(Y)$ is either $\mathbb{C}$-linear or $\mathbb{C}$-antilinear (and can therefore be described readily by using Theorem 1). That result is proved by Royden’s method of analysing the shape of the unit spheres. The same method is used in [2] and [3] to obtain our Theorem 1. The simpler method we use here does not appear to handle the $\mathbb{R}$-linear case. The surjective $\mathbb{R}$-linear isometries between $Q(X)$ and $Q(Y)$ for general Riemann surfaces have not yet been classified.

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References

[1] S. BANACH, *Theory of Linear Operations*, trans. F. Jellett, North-Holland Math. Library **38**, North-Holland, Amsterdam, 1987. MR 88a:01065

[2] C. J. EARLE and I. KRA, “On holomorphic mappings between Teichmüller spaces” in *Contributions to Analysis*, Academic Press, New York, 1974, 107 – 124. MR 55:3324

[3] ———, *On isometries between Teichmüller spaces*, Duke Math. J. **41** (1974), 583 – 591. MR 50:596

[4] O. FORSTER, *Lectures on Riemann Surfaces*, trans. B. Gilligan, Grad. Texts in Math. **81**, Springer, New York, 1991. MR 93h:30061

[5] N. LAKIC, *An isometry theorem for quadratic differentials on Riemann surfaces of finite genus*, Trans. Amer. Math. Soc. **349** (1997), 2951 – 2967. MR 97i:30062

[6] J. LAMPERTI, *On the isometries of certain function–spaces*, Pacific J. Math. **8** (1958), 459 – 466. MR 21:3764

[7] V. MARKOVIC, *Biholomorphic maps between Teichmüller spaces*, Duke Math. J. **120** (2003), 405 – 431.

[8] S. NAG, *The Complex Analytic Theory of Teichmüller Spaces*, Canad. Math. Soc. Ser. Monogr. Adv. Texts, Wiley, New York, 1988. MR 89f:32040

[9] H. L. ROYDEN, "Automorphisms and isometries of Teichmüller space" in *Advances in the Theory of Riemann Surfaces (Stony Brook, N.Y., 1969)*, Ann. of Math. Stud. **66**, Princeton Univ. Press, Princeton, 1971, 369 – 383. MR 44:5452

[10] ———, *Real Analysis*, 3rd ed., Macmillan, New York, 1988. MR 90g:00004

[11] W. RUDIN, *L^p-isometries and equimeasurability*, Indiana Univ. Math. J. **25** (1976), 215 – 228. MR 53:14105

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